Estimation of Lyapunov spectra from space-time data

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A method to estimate Lyapunov spectra from spatio-temporal data is presented, which is well-suited to be applied to experimental situations. It allows to characterize the high-dimensional chaotic states, with possibly a large number of positive Lyapunov exponents, observed in spatio-temporal chaos. The method is applied to data from a coupled map lattice.

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Nonlinear spatially-extended systems (SES) attracted a vast research activity in nonlinear dynamics, partly because of their ability to exhibit high-dimensional chaotic motion, with possibly a large number of positive Lyapunov exponents (for an overview on SES see [1]). Established dynamical models for SES are partial differential equations (PDE) for systems continuous in space and time, and coupled map lattices (CML) for systems discrete in space and time. The spectrum of Lyapunov exponents can be directly estimated from the models by numerical means. On the basis of the Lyapunov spectrum the Kaplan-Yorke dimension and the metric entropy can be estimated. Along these lines, an extensive scaling of the Lyapunov spectrum, as well as the Kaplan-Yorke dimension and the metric entropy, with respect to the system size has been observed for several PDE- and CML-models [2]. The term 'spatio-temporal chaos' has been coined to describe this phenomenology. Additionally, various well-controlled experiments to investigate the nature of spatio-temporal chaos were conducted. Unfortunately, it was not possible to directly extract the Lyapunov spectrum of the experimental systems from the space-time data so far, since one has to find an appropriate nonlinear model first. Typically, this is done by using a time-delayed embedding (of high enough order) to construct a phase space from a scalar measurement [3]. Then, the time-dependent Jacobian can be estimated from the data. Unfortunately, one faces two problems for relatively high dimensions of phase space (i. e. $D \geq 6$): (1) An exponentially increasing number of data is required. (2) The delay embedding induces a folding in phase space, which makes it increasingly difficult to perceive a deterministic structure

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[4]. For the above reasons the established techniques for the estimation of Lyapunov spectra are restricted to chaotic states with a small number of positive Lyapunov exponents (say N^+ smaller or equal to 3). Besides that spatio-temporal chaos was characterized by the decomposition into spatial modes such as the Karhunen-Loeve decomposition [5,6].

In this paper we address the question of how to estimate the spectrum of Lyapunov exponents of a SES from space-time data, which could i.e. be the output of an experiment. Typically, the equation of motion is unknown. Recently, novel methods for modelling SES on the basis of models with a purely local coupling (PDE, CML) were established [7–9]. We show how the latter local models can be used to estimate the Lyapunov spectrum of a SES from space-time data. Then, we apply the method to space-time data of a CML and compare the results to the theoretical prediction. Finally, we show how the method can be used to investigate boundary effects and the 'universal' nature of the Lyapunov spectrum of a homogeneous SES.

Let us recall the definition of the spectrum of Lyapunov exponents of a non-autonomous dynamical system

$$\vec{x}_{n+1} = \vec{f}(\vec{x}_n, \vec{y}_n), \tag{1}$$

where $\vec{x}_n \in \mathcal{R}^I, \vec{y}_n \in \mathcal{R}^J$ is an external driving force, and $\vec{f} : \mathcal{R}^{I+J} \rightarrow \mathcal{R}^I$. For the ease of presentation we restrict the following discussion on time-discrete systems. The spectrum of Lyapunov exponents $\lambda_i, i = 1, ..., I$ of the system (1) is defined as the contraction rates of a volume element averaged over the attractor [3,10]:

$$\lambda_i(\vec{x}_0, \vec{y}_n) = \lim_{T\rightarrow\infty} \frac{1}{T} \ln |\Lambda_i^{(T)}|, \tag{2}$$

where $\Lambda_i^{(T)}$ are the eigenvalues of the product of all Jacobians $\mathbf{J}_n = \frac{\delta \vec{f}}{\delta \vec{x}}(\vec{x}_n, \vec{y}_n)$:

$$\prod_{n=1}^{T} \mathbf{J}_n \vec{u}_j^{(T)} = \Lambda_i^{(T)} \vec{u}_i^{(T)}. \tag{3}$$

In general, the spectrum depends on the initial state $\vec{x}_0$, and the sequence of inputs $\vec{y}_n$. In special cases, these dependences can be removed with the help of multiplicative ergodic theorems. It has been shown that in the autonomous case ($\vec{y}_n$=0), as well as for random inputs $\vec{y}_n$ the limit (2) exists and only depends on the invariant measure [3,10]. In basically all relevant cases the Lyapunov spectrum λ_i cannot be computed from the equations (1)- (3) directly and one has to rely on numerical estimations of the Lyapunov spectrum. A well-established technique for its estimation from time-evolution equations (1) is described in [3]: The trajectory as well as I perturbation vectors are iterated numerically (with the help of the time-dependent Jacobian $\mathbf{J}_n$). A successive orthonormalization of the perturbation vectors prevents them from alignment. From the Lyapunov spectrum important invariant quantities can be estimated: 1) The Kaplan-Yorke dimension, $D_{KY} = k + \frac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|}$, where $\sum_{i=1}^{k} \lambda_i \geq 0$ and $\sum_{i=1}^{k+1} \lambda_i < 0$. 2) An upper bound for the metric entropy via the Pesin formula, $h = \sum_{i=1}^{N^+} \lambda_i$, where N^+ is the number of positive Lyapunov exponents.

At first we concentrate for simplicity on discrete, scalar systems, with a local coupling only, which are modeled with the help of coupled map lattices (CML). The cases of continuous dynamics in space and time, as well as multi-component systems can be treated

analogously and will be discussed at the end of the paper. Let the experimental data $\{x_n^i\}$ be a measurement of a homogeneous, scalar spatially-extended system with a discrete spatial variable $i = 1, ..., I$ and a discrete time $n = 1, ..n = 1, .., N$. The overall number of data is NI. If the unknown equation of motion of the system under investigation is a CML with a local coupling of range m, the data can be modeled by

$$x_{n+1}^i = \hat{h}(\vec{v}_n^i), \tag{4}$$

with $\vec{v}_n^i = (x_n^{i-m}, \ldots, x_n^{i+m})$ and $\hat{h} : \mathcal{R}^{2m+1} \to \mathcal{R}$, for $i = m+1, .., I-m$. The system (4) is non-autonoumous, since it is driven by the variables at the boundaries $(x_n^1, \ldots, x_n^m, x_n^{I-m}, \ldots, x_n^I)$. The phase space of such a system has $(N-2m)$ dimensions, allowing for a high-dimensional chaotic motion with possibly a lot of positive Lyapunov exponents. Nevertheless, the restricted couplings of the variables in phase space - as it is expressed by the function $\hat{h}$, which couples only four values of the space-time data - allows to obtain a nonlinear, deterministic model in a $(2m+2)$-dimensional space only. It is of fundamental importance to statistically verify the model class (4) on the basis of the data with appropriate tools, which also allow to estimate an appropriate coupling range m. Since this has been described in detail elsewhere, we refer to the literature [7–9].

In this paper we use a local linear model $\hat{h}(\vec{v}_n^i) = \vec{a}_n^i \vec{v}_n^i + b_n^i$, where the parameters $(\vec{a}_n^i, b_n^i)$ can be obtained for each time $n, n = 2, ..., N$ and each position $i, i = m+1, ..., I-m$ by the minimisation of the distance

$$\sigma_n^i = \sqrt{\frac{1}{N_{U_n^i}} \sum_{\vec{v}_n^i \in U_n^i} \left(x_{n+1}^i - \hat{h}(\vec{v}_n^i) \right)^2}, \tag{5}$$

of the data to the local linear model with the help of a local least squares fit. U_n^i is a sufficiently small neighbourhood of $\vec{v}_n^i$ with the cardinal number $N_{U_n^i}$. Since the model $\hat{h}$ is homogeneous, appropriate neighbors can be chosen with respect to a varying time n as well as a varying position i. This allows to treat every space-time point x_n^i (except at the boundaries) on the same basis and to vastly enhance the statistics and decrease the requirements on the number of data by averaging over space and time. The local linear fits have been chosen for practical purposes; we note that in principle any appropriate method for nonlinear fitting can be used.

As a consequence of the local coupling, the Jacobian $\mathbf{J}_n(x_n^1, \ldots, x_n^m, x_n^{I-m}, \ldots, x_n^I)$ of a SES has the form of a $(2m+1)$-diagonal matrix, which in the case of a CML-model (4) reads,

$$\mathbf{J}_n^{(ij)} = \begin{cases} 0, & \text{if } |i-j| > m, \\ \frac{\partial \hat{h}}{\partial x_n^j}(\vec{v}_n^i), & \text{else,} \end{cases} \tag{6}$$

for $i, j = m+1, .., I-m$. In the case of a local linear fit for $\hat{h}$, the non-zero entries of the Jacobian (6) are given by the coefficients $\vec{a}_n^i$. Let us recall that the spectrum of Lyapunov exponents (2) depends on the sequence of inputs, which in the case of SES are the boundaries. While in priniciple, this dependence on the sequence of inputs at the boundaries cannot be removed, we found some evidence that if the investigated system is part of a larger system, which displays homogeneous spatio-temporal chaos, the Lyapunov exponents do not depend on the sequence of inputs at the boundaries. This will be discussed in more detail later.

We performed several tests of the above presented method for established models of SES. At first, we present results on the estimation of Lyapunov spectra for a lattice of coupled logistic maps

$$x_{n+1}^i = (1-2\epsilon)f(x_n^i) + \epsilon(f(x_n^{i-1}) + f(x_n^{i+1})), \tag{7}$$

with $f(x) = 4x(1-x)$, from space-time data $\{x_n^i\}, n = 1, ..., 10,000, i = 1, ..., 50$ with fixed boundary conditions $x_n^1 = x_n^I = 0.1$ and $\epsilon = 0.2$. In this case, since the boundary conditions are fixed, the system is autonomous with a 48-dimensional phase space. The first step is to identify the data $\{x_n^i\}$ to be governed by a local model (4) and to estimate the appropriate value of the coupling range m. Since the estimation and verification of local models from space-time data has been already described in detail, we refer to the literature [7–9], and from now on consider the model (4) to be verified. The function $\hat{h}$ was modelled with a local linear model under varying m, the neighborhoods U_n^i were chosen such that at least 30 points were contained. The coefficients of the Jacobian $\mathbf{J}_n$ were extracted according to (6). We propagated $(50-2m)$ perturbation vectors in time according to the fitted Jacobian $\mathbf{J}_n$ and orthonormalized the perturbation vectors after each time step. The spectrum of the Lyapunov exponents was computed as a mean over the logarithms of the stretching factors of the propagation vectors (for details see [3]). We compare the results to the estimation of the Lyapunov spectrum from the equations (7) As presented in Fig. 1. we find a good agreement of the spectra for $m \geq 1$. The estimation of the exponents for $m = 0$ does not yield a meaningful result and therefore the $(m = 0)$-model has to be rejected. Since the Lyapunov spectrum under variation of the coupling range m converges for $m \geq 1$, we propose the convergence of the LS as another criterion for estimating m. In Table 1 we compare the dimension, the entropy and the maximum Lyapunov for $m = 0, 1, 2$.

Our approach of a non-autonomous model (4) allows to estimate Lyapunov spectra not only in the case in which all variables $i = 1, ..., I$ are observed, but also for incomplete observations in space $i = i_0, ..., i_0+l-1$. The so-defined subsystems of length l are driven by the input sequences at the boundaries $(x_n^{i_0}, \ldots, x_n^{i_0+m}, x_n^{i_0+l-m}, \ldots, x_n^{i_0+l-1})$, where in general the Lyapunov spectrum also depends on. Therefore, the dimension of the phase space is $l-2m$ and the dimension density δ is calculated as $\delta = \frac{D_{KY}}{l-2m}$, and the entropy density η is calculated as $\eta = \frac{H}{l-2m}$.

At first, we present results of the estimation of the dimension density δ and the entropy density η for such a subsystem under variation of the length of the subsystem l in Fig. 2. The data were taken from a CML with $\epsilon = 0.3$ and the total length $I = 100$, starting at i_0=40. For the estimation of the Lyapunov spectrum an observation time $N = 20,000$ was taken into account. We focus on a model with $m = 1$. One finds a convergence of the dimension and the entropy for larger values of l, as one expects for extensive systems.

Furthermore, we estimate Lyapunov spectra of subsystems $\{x_n^i\}, i = i_0, ..., i_0 + l - 1$ of the SES (4) with a varying spatial offset i_0 and length $l = 10$. With this, we unveil the effects of boundaries as well as the independence of the spectrum sufficiently far away from the boundaries. The latter effect can be used to estimate the spectrum of a large SES from a small subsystem only [11,12] . The subsystem of length $l = 10$ is taken from a CML (7) of length $(I = 100)$ with fixed boundary conditions and $\epsilon = 0.3$. For the estimation an observation time $N = 20,000$ was taking into account. We find that the Lyapunov exponents approach limit values for large enough offset i_0, while the values of the exponents

are significantly decreased in the vicinity of the boundary. From the spectra the dimension density $\delta = D_{KY}/l$ and the entropy density $\eta = h/l$, where h is the metric entropy, are estimated as shown in Fig. 3. For $i_0 > 5$ we observe a plateau of the dimension and the entropy in the limits of the statistical errors of the estimations. For $i_0 \leq 5$, both quantities are decreased as an effect of the fixed boundaries of the system. While the dynamics as well as the input sequence at the boundaries of all the subsystems are different for different spatial offsets i_0, we find a 'universal' Lyapunov spectrum sufficiently far away from the boundaries. We take the latter results as an evidence that the Lyapunov spectra of the non-autonomous subsystems are independent of the external driving at their boundaries in this case. Therefore, the Lyapunov-spectra of a subsystem of a homogeneous SES, sufficiently far away from the boundaries only depends on the size of the subsystem and is an appropriate tool to characterize the dynamics of the subsystem as well as the larger system itself as has been conjectured by Grassberger [13].

Until now, we described the technique to estimate Lyapunov spectra of SES from space-time data on a rather fundamental level. In the following we shortly comment on more general cases. A more profound discussion will be presented in a subsequent paper. The next step is to also consider CMLs in 2 and 3 spatial dimensions: The increase in spatial dimensions immediately implies an increase of the number of nearest neighbors. Additionally, the structure of the Jacobian will be such that there are several diagonals with non-zero entries, which are separated by diagonal bands of zeros. Also the investigation of non-homogeneous systems must be taken into account. In this case the model in eq. (4) depends on the space, $\hat{h} = \hat{h}_i$, as well as the partial derivatives of the model in eq. (6). A non-homogeneous model $\hat{h}_i$ does not introduce any conceptual problems, but a larger number of data is required since the average over space and time for the fitting of the model as applied in our example, cannot be applied for a non-homogeneous model. For practical purposes it is by far more interesting to estimate the Lyapunov spectrum of SES continuous in space and time as modelled by PDEs. The identification of PDEs from experimental data involves the estimation of derivatives with respect to space and time as shown in [7,9]. For the estimation of the Lyapunov spectrum though, one has to introduce some appropriate discretization in space and time, as it is common in the case of the estimation of Lyapunov spectra from a PDE directly. After the discretization, the procedure evolves along the same lines as described for CMLs, where the model $\hat{h}$ of the time-evolution equation (4) takes the form of a discretized PDE, while an additional dependence of the Lyapunov spectra on the discretization in space and in time has to be taken into account. In the case of a N-component CML or PDE with $N > 2$ (the state space dimension is N), the measurement of N-component space-time data is required for the estimation of the Lyapunov spectrum. The important problem of how to identify and characterize a N-component SES from a scalar space-time measurement is subject to current research.

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[1] P. C. Cross, M. C. Hohenberg, Rev. Mod. Phys. **65** (1993) 851.
[2] D. Ruelle, Comm. Math. Phys. **87** (1982) 287.
[3] H. Kantz and T. Schreiber, *Nonlinear time series analysis* (Cambridge University Press, Cambridge, UK, 1997).
[4] E. Olbrich and H. Kantz, Phys. Lett. A **232** (1997) 63.
[5] A. Torcini, A. Politi, G.P. Puccioni, G. D'Alessandro, Physica D, **53** (1991) 85.
[6] S. M. Zoldi, and H. S. Greenside, Phys. Rev. Lett. **78** (1997) 1687.
[7] H. Voss, M. J. Bünner, and M. Abel, Phys Rev. E **57** (1998) 2820.
[8] U. Parlitz, *Nonlinear Time-Series Analysis*, in: *Nonlinear Modeling - Advanced Black-Box Techniques*, Eds. J.A.K. Suykens and J. Vandewalle, Kluwer Academic Publishers, Boston, (1998) p. 209-239.
[9] M. Bär, R. Hegger H. Kantz, Phys. Rev. E **59** (1999) 337.
[10] M. Casdagli, in *Nonlinear Modeling and Forecasting, SFI Studies in the Sciences of Complexity* (Addison-Wesley, Reading, MA, 1992).
[11] R. Carretero-González, S. Orstavik, J. Huke, D. S. Broomhead, J. Stark, preprint server chaodyn/9807038.
[12] N. Parekh, V. R. Kumar, and B. D. Kulkarni, Chaos **8** (1998) 300.
[13] P. Grassberger, Physica Scripta **40** (1989) 346.

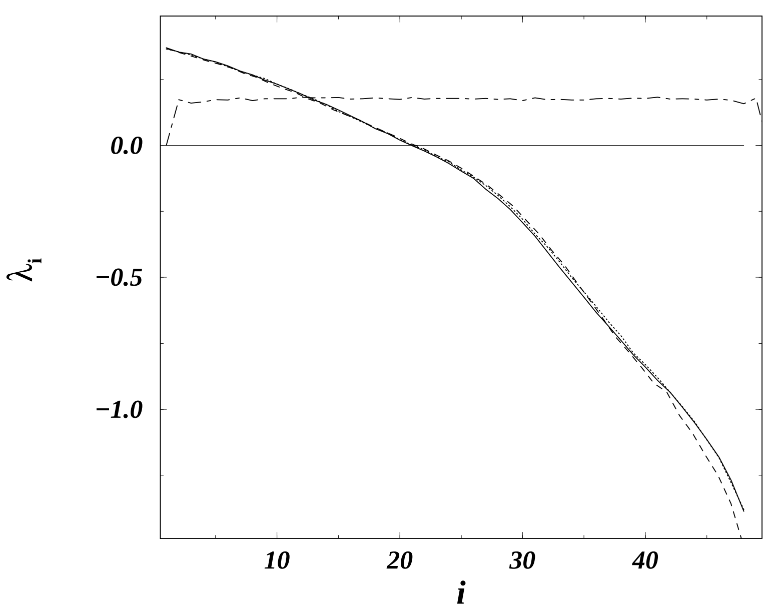

FIG. 1. Lyapunov spectrum of a coupled map lattice of 50 sites estimated from the equations (solid line), and from space-time data with $m = 0$ (dot-dashed line; divided by 10 for presentational purposes), $m = 1$ (dotted line), and $m = 2$ (dashed line).

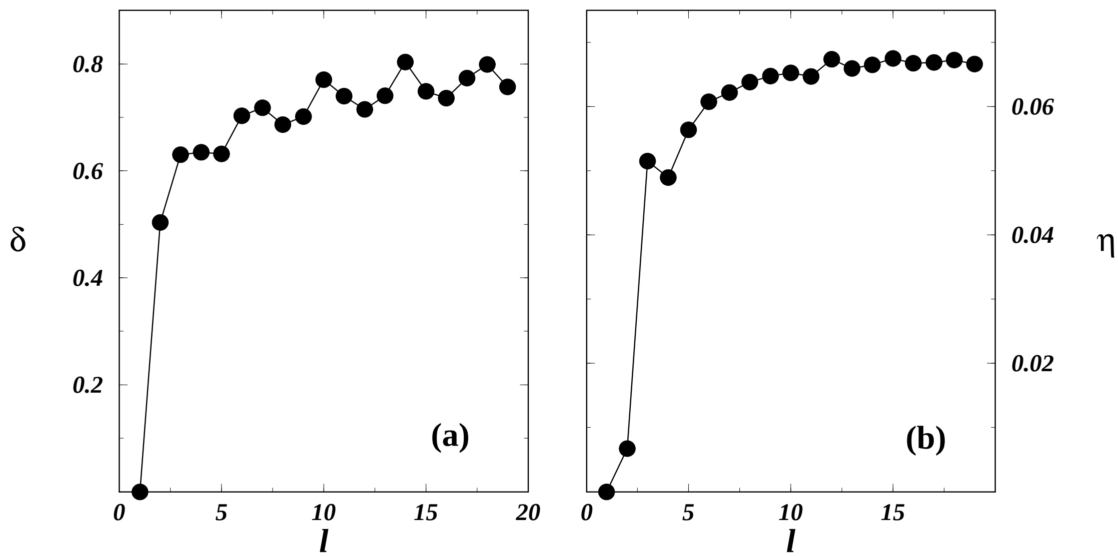

FIG. 2. (a) Dimension density δ, and (b) entropy density η as estimated from a system of total length $I = 100$ (at the site $i_0 = 40$) with varying length l of the subsystem.

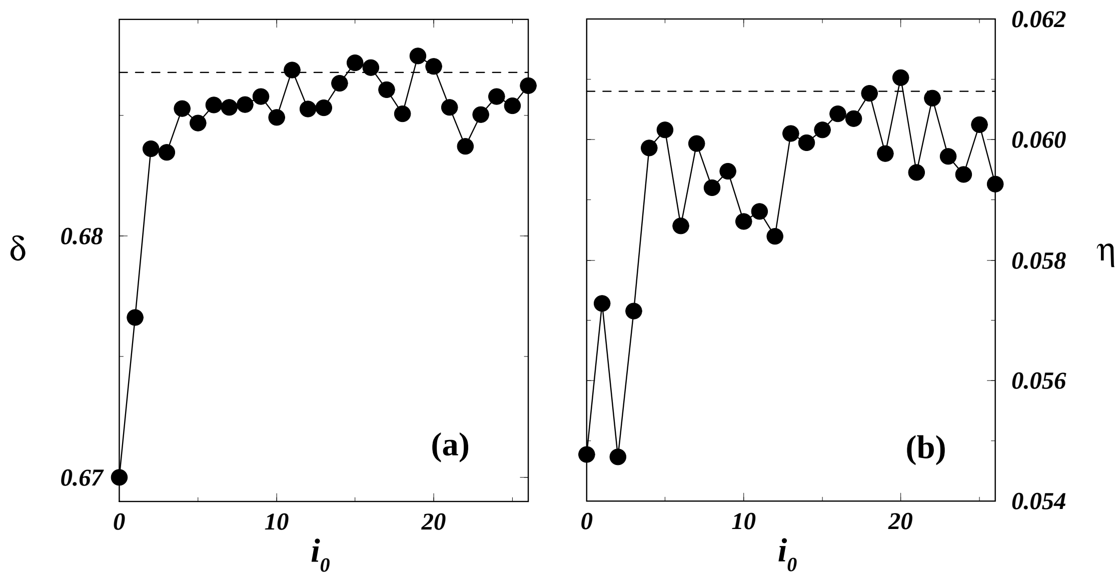

FIG. 3. (a) Dimension density δ, and (b) entropy density η as estimated from a subsystem of length $l = 10$ with varying distance i_0 from the boundaries. The horizontal lines are the values for the densities taken at $l = 8$ according to Fig. 2.

	δ	η	λ_{max}
equation	0.72	0.085	0.37
data ($m = 0$)	–	3.4	3.6
data ($m = 1$)	0.72	0.085	0.37
data ($m = 2$)	0.70	0.085	0.37

TABLE I. Estimation of the dimension density δ, the entropy density η, and the largest Lyapunov exponent λ_{max} from the equations (first row), and the from the data (second row).